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EVALUATION OF INTEGRAL OF ELLIPTIC GAUSSIAN DISTRIBUTION OVER A CENTERED ELLIPSE

by

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ABSTRACT. The integral of the elliptic Gaussian distribution over a centered ellipse may be evaluated after simple preliminary transformations from tables of the circular coverage function. The probable value of the distribution over a centered but randomly oriented ellipse where all orientations are equally probable may be evaluated from an integral involving the circular coverage function.

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AN ACTIVITY OF THE BUREAU OF NAVAL WEAPONS

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FOREWORD

The estimation of the probability of missiles falling within specified areas is important in their development and application. For certain applications the accurate evaluation of this probability is required.

This report presents some techniques for accurately evaluating from tables of the circular coverage function, the probability of a missile hit within a centered elliptical area under various assumed conditions where the missile deviations are normally distributed.

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INTRODUCTION

The estimation of the probability of a missile falling within a specified area under various assumed conditions has been the subject of study for many years. In general, it is assumed that the range and lateral deviations of the missile are normally distributed (i. e., Gaussian-distributed).

Simple cases are those of a rectangular target area with sides parallel to range and lateral dispersions (assumed uncorrelated), and a circular target area centered with respect to a circular Gaussian distribution. Recently, other cases such as the estimation of the probability of a missile falling within a circular target area where the distribution in missile deviations is offset, circular or elliptical Gaussian, has received study, especially by the Rand Corp., Santa Monica, California (Ref. 1 to 12).

In general, two methods are available for estimating the probability of a missile hit. One method is graphical and consists in plotting the suitably transformed target area on circular probability paper and summing the probability values of the elements within the target area. The other method consists in evaluating the probability from tabulated functions. The relative convenience of the methods will depend upon the nature of the problem involved. However, it should be pointed out that summing probability elements can be fatiguing and the probability of a missile hit can be evaluated more accurately from tabulated functions. In many cases such accuracy is not required, but in cases where the number of missiles thrown is large, and the individual probabilities of success are small, the accuracy of tabulated functions may be required.

DISCUSSION

In this report some techniques involving the circular coverage function are discussed for estimating the probability of a hit in the use of missiles against a target under the following assumptions:

1. The missile range and lateral deviations are normally distributed and known.
2. The target is a circle or an ellipse.
3. The center of the target and center of impact are coincident.
4. The orientation of the target with respect to the distribution is known or it is random, all orientations being equally probable.

In order to avoid unnecessary discussion of well-known familiar transformations, reduction to standard form (i. e., normalization) will be assumed wherever feasible.

CIRCULAR COVERAGE FUNCTION

The circular coverage function (Ref. 8) is the probability that a missile will hit a circle of radius R if it is aimed at a point a distance r from the center of the circle and if it is subject to a circular Gaussian impact-probability law of unit standard deviation. Reference 8 does not give a derivation of this function, but it may be derived as follows.

In Fig. 1, let the center of impact be represented by the origin O , and let the target be represented by the circle with center C and radius R . Without loss in generality an x, y -coordinate system with origin at O such that C lies on the positive x -axis may be assumed. Then the probability $p(R, r)$ that the missile will hit the circle is given by

$$(1) \quad p(R, r) = \frac{1}{2\pi} \iint_{(x-r)^2 + y^2 \leq R^2} e^{-(x^2+y^2)/2} dx dy$$

Assuming polar coordinates t, θ with the pole at C and the polar axis along the positive direction of the x -axis

$$(2) \quad \begin{aligned} x &= r + t \cos \theta \\ y &= t \sin \theta \end{aligned}$$

and Eq. 1 becomes

$$(3) \quad p(R, r) = \frac{1}{2\pi} e^{-(r^2/2)} \int_0^R e^{-(t^2/2)} t \int_{-\pi}^{\pi} e^{(-rt \cos \theta)} d\theta dt$$

From Bessel function theory (Ref. 13)

$$(4) \quad e^{(-rt \cos \theta)} = I_0(rt) + 2 \sum_{n=1}^{\infty} (-1)^n I_n(rt) \cos n\theta$$

Substituting Eq. 4 into Eq. 3 and integrating

$$(5) \quad p(R, r) = e^{-(r^2/2)} \int_0^R e^{-(t^2/2)} I_0(rt) t dt$$

Equation 5 is the circular coverage function. Certain special cases and properties of this function should be noted.

$$(6) \quad p(R, 0) = 1 - e^{-(R^2/2)} \quad \text{Circular Gaussian probability distribution}$$

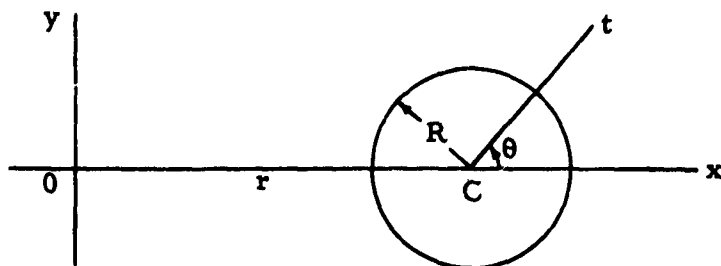


FIG. 1. Offset Circular Target.

$$(7) \quad \underline{p}(R, \infty) = 0 \quad \text{Target infinitely far from center of impact}$$

$$(8) \quad \underline{p}(0, r) = 0 \quad \text{Target of null area}$$

$$(9) \quad \underline{p}(\infty, r) = 1 \quad \text{Target of infinite area}$$

$$(10) \quad \underline{p}(-a, \beta) = \underline{p}(a, -\beta) = \underline{p}(-a, -\beta) = \underline{p}(a, \beta)$$

Equations 6, 7, 8, and 10 follow directly from Eq. 5. Equation 9 may be obtained by substituting (Ref. 13)

$$(11) \quad I_0(rt) = \sum_{k=0}^{\infty} \frac{(rt)^{2k}}{2^{2k}(k!)^2}$$

in the integrand of Eq. 5 and integrating term by term.

Tables of the circular coverage function are given in Ref. 8. Extensive tables (Ref. 9) and abridged tables (Ref. 11) are available in which the function

$$(12) \quad \underline{q}(a, \beta) = e^{[-(a^2/2)]} \int_{\beta}^{\infty} e^{[-(V^2/2)]} I_0(aV) V dV$$

is tabulated. Since from Eq. 5, 9, and 12

$$(13) \quad \underline{p}(R, r) = 1 - \underline{q}(R, r)$$

these tables may be used to obtain $\underline{p}(R, r)$. The function $\underline{q}(R, r)$ is the probability that the missile will not hit the target.

INTEGRAL OF THE CIRCULAR GAUSSIAN DISTRIBUTION OVER A CENTERED ELLIPSE

The general form of this integral is given by

$$(14) \quad p(a, b) = \frac{1}{2\pi} \iint_{(x^2/a^2 + y^2/b^2) \leq 1} e^{-(x^2 + y^2)/2} dx dy$$

In what follows the elliptic semiaxes a and b are assigned their positive values. Under the transformation

$$(15) \quad \begin{aligned} x &= t \cos \theta \\ y &= t \sin \theta \end{aligned}$$

Equation 14 becomes

$$(16) \quad p(a, b) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{t(\theta)} e^{-(t^2/2)} t dt d\theta$$

where

$$(17) \quad t(\theta) = \frac{ab}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}}$$

Integrating Eq. 16 with respect to t and substituting Eq. 17 for the upper limit of integration

$$(18) \quad \begin{aligned} p(a, b) &= 1 - \frac{1}{2\pi} \int_0^{2\pi} e^{-(1/2)ab^2/(a^2 \sin^2 \theta + b^2 \cos^2 \theta)} d\theta \\ &= 1 - \frac{2}{\pi} \int_0^{\pi/2} e^{-(1/2)ab^2/(a^2 \sin^2 \theta + b^2 \cos^2 \theta)} d\theta \end{aligned}$$

due to the periodicity of the integrand. Applying the transformation

$$(19) \quad \tan \theta = \frac{b}{a} \tan \frac{\zeta}{2}$$

to Eq. 18,

$$(20) \quad p(a, b) = 1 - \frac{1}{\pi} e^{-(a^2+b^2)/4} \int_0^\pi e^{-[(a^2-b^2)/4]\cos \zeta} \frac{2abd\zeta}{a^2+b^2+(a^2-b^2)\cos \zeta}$$

In order to evaluate Eq. 20, the integrand will be transformed into a tractable form. From Bessel function theory (Ref. 13).

$$(21) \quad e^{-[(a^2-b^2)/4]\cos \zeta} = I_0\left(\frac{a^2-b^2}{4}\right) + 2 \sum_{n=1}^{\infty} (-1)^n I_n\left(\frac{a^2-b^2}{4}\right) \cos n\zeta$$

Now the function

$$(22) \quad \frac{1-z}{1+z} = 1 + 2 \sum_{m=1}^{\infty} (-1)^m z^m; \quad |z| < 1$$

is considered. Substituting

$$(23) \quad z = \mu e^{i\zeta}; \quad |\mu| < 1$$

in Eq. 22, expanding and equating real terms

$$(24) \quad \frac{1 - \mu^2}{1 + \mu^2 + 2\mu \cos \zeta} = 1 + 2 \sum_{m=1}^{\infty} (-1)^m \mu^m \cos m\zeta$$

Substituting

$$(25) \quad \mu = \frac{a - b}{a + b}; \quad |\mu| < 1 \quad (\text{since } a \text{ and } b > 0)$$

in Eq. 24 and clearing fractions

$$(26) \quad \frac{2ab}{a^2 + b^2 + (a^2 - b^2) \cos \zeta} = 1 + 2 \sum_{m=1}^{\infty} (-1)^m \left(\frac{a - b}{a + b} \right)^m \cos m\zeta$$

Substituting Eq. 21 and 26 for the integrand in Eq. 20

$$(27) \quad p(a, b) = 1 - \frac{1}{\pi} e^{[-(a^2+b^2)/4]} \int_0^\pi \left[I_0 \left(\frac{a^2 - b^2}{4} \right) + 2 \sum_{n=1}^{\infty} (-1)^n I_n \left(\frac{a^2 - b^2}{4} \right) \cos n\zeta \right] \\ \times \left[1 + 2 \sum_{m=1}^{\infty} (-1)^m \left(\frac{a - b}{a + b} \right)^m \cos m\zeta \right] d\zeta \\ = 1 - e^{[-(a^2+b^2)/4]} \left[I_0 \left(\frac{a^2 - b^2}{4} \right) + 2 \sum_{n=1}^{\infty} \left(\frac{a - b}{a + b} \right)^n I_n \left(\frac{a^2 - b^2}{4} \right) \right]$$

It remains to express $p(a, b)$ in terms of the circular coverage function. Now the circular coverage function (Eq. 5) can be expressed in the following forms

$$(28) \quad p(R, r) = e^{[-(r^2/2)]} \int_0^R e^{[-(t^2/2)]} I_0(rt) t dt \\ = 1 - e^{[-(r^2/2)]} \int_R^\infty e^{[-(t^2/2)]} I_0(rt) t dt \\ = e^{[-(R^2+r^2)/2]} \sum_{n=1}^{\infty} \left(\frac{R}{r} \right)^n I_n(Rr) \\ = 1 - e^{[-(R^2+r^2)/2]} \sum_{n=0}^{\infty} \left(\frac{r}{R} \right)^n I_n(Rr)$$

The latter two forms of Eq. 28 are obtained from the previous forms by repeated integration by parts, making use of the relations (Ref. 13).

$$(29) \quad I_{-n}(t) = I_n(t); \text{ (n an integer)}$$

$$\frac{d}{dt} [t^n I_n(t)] = t^n I_{n-1}(t)$$

$$\frac{d}{dt} [t^{-n} I_n(t)] = t^{-n} I_{n+1}(t)$$

Now let

$$(30) \quad R = \frac{a - b}{2}$$

$$r = \frac{a + b}{2}$$

Then from Eq. 27, 28, and 30

$$(31) \quad p(a, b) = 1 - e^{-(a^2+b^2)/4} \left[I_0 \left(\frac{a^2 - b^2}{4} \right) + 2 \sum_{n=1}^{\infty} \left(\frac{a - b}{a + b} \right)^n I_n \left(\frac{a^2 - b^2}{4} \right) \right]$$

$$= p \left(\frac{a + b}{2}, \frac{a - b}{2} \right) - p \left(\frac{a - b}{2}, \frac{a + b}{2} \right)$$

From Eq. 13 and 31

$$(32) \quad p(a, b) = p \left(\frac{a + b}{2}, \frac{a - b}{2} \right) - p \left(\frac{a - b}{2}, \frac{a + b}{2} \right)$$

$$= q \left(\frac{a - b}{2}, \frac{a + b}{2} \right) - q \left(\frac{a + b}{2}, \frac{a - b}{2} \right)$$

Therefore the probability, $p(a, b)$, of a missile hitting the centered elliptical target with semiaxes a and b can be computed from available tables (Ref. 8, 9, and 11). It will be noted that if $a = b = R$, the target is a circle and Eq. 31 and 32 reduce to Eq. 6, the circular probability distribution. From Eq. 10 and 13 it is obvious that the absolute value of $a - b$ may be used in Eq. 31 and 32.

From a geometrical point of view, the integral, $p(a, b)$, of the Gaussian distribution over a centered ellipse is equal to the difference

of the integrals of the distribution over two associated offset circles as given by Eq. 32. A centered ellipse with associated offset circles A and B is shown plotted on a circular probability grid in Fig. 2. By Eq. 32, the integral of the distribution over the ellipse is equal to the integral of the distribution over the lune bounded by the offset circles.

A comparison of the values of $p(a, b)$ for the ellipse in Fig. 2, obtained by interpolation from the tables of Ref. 8 and by count of probability elements, is of interest. The semiaxes of the ellipse are $a = 1.690$ and $b = 0.845$ standard deviations, respectively. Then $(a + b)/2 = 1.268$ and $(a - b)/2 = 0.423$. The values obtained by both methods are given in Table 1. The results show good agreement.

TABLE 1. Evaluation of the Integral of the Circular Gaussian Distribution Over a Centered Ellipse From Tables and by Probability Element Count

Area of Integration	Value of the Integral	
	From Tables of Ref. 8	By Probability Element Count
Offset circle A $p(1.268, 0.423)$	0.5206	0.520
Offset circle B $p(0.423, 1.268)$	0.0397	0.041
Lune $p(1.268, 0.423) - p(0.423, 1.268)$	0.4809	0.479
Ellipse $p(1.690, 0.845)$	0.4809 ^a	0.478

^aIntegral over ellipse equal to that over lune by Eq. 32.

INTEGRAL OF THE ELLIPTIC GAUSSIAN DISTRIBUTION OVER A CENTERED CIRCLE

The general form of this integral is given by

$$(33) \quad p(R, \sigma_x, \sigma_y) = \frac{1}{2\pi\sigma_x\sigma_y} \iint_{x^2+y^2 \leq R^2} e^{-(x^2/2\sigma_x^2 + y^2/2\sigma_y^2)} dx dy$$

Under the transformation

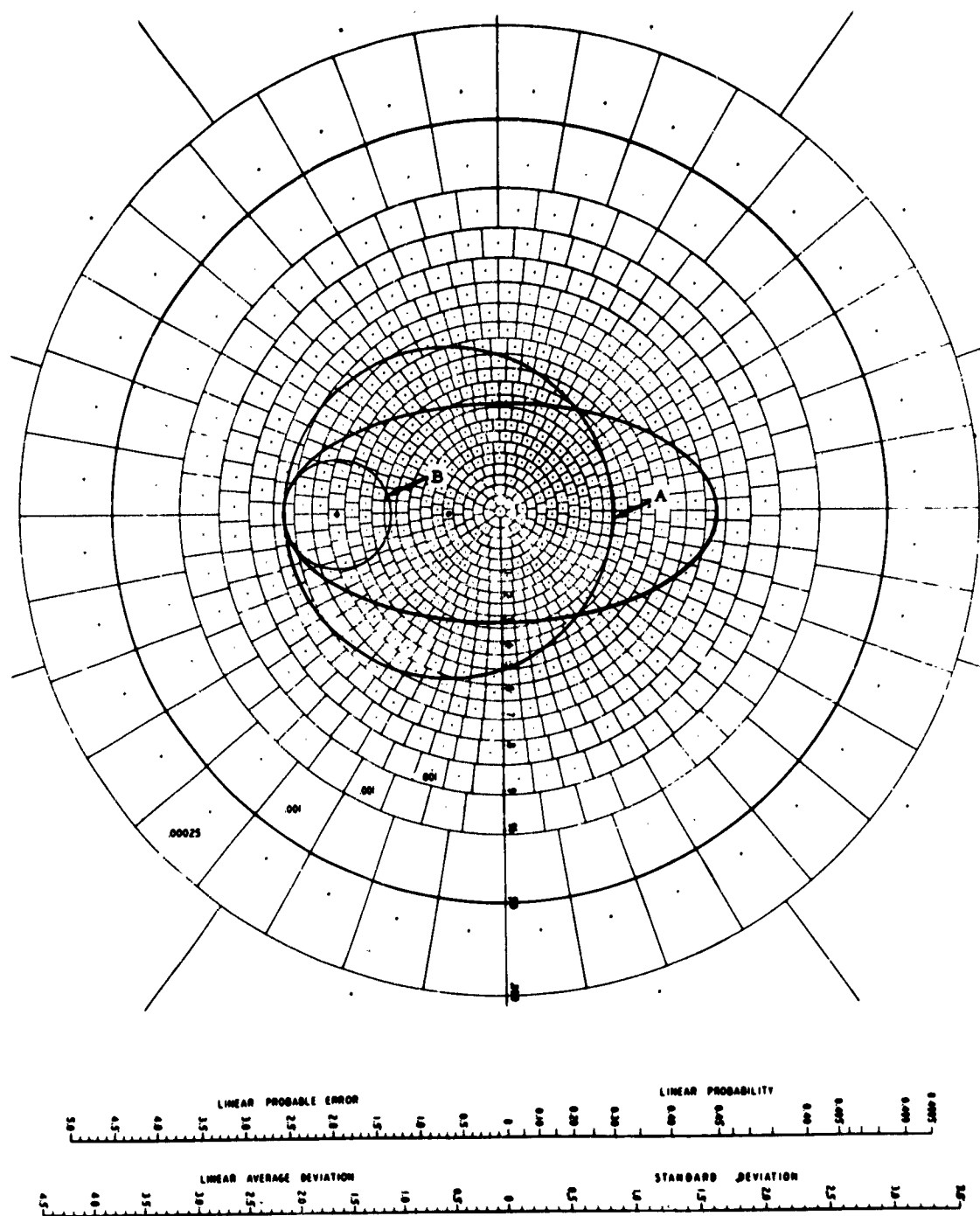


FIG. 2. Centered Ellipse With Associated Offset Circles Plotted on a Circular Gaussian Distribution Grid.

$$(34) \quad \begin{aligned} x &= \sigma_x u \\ y &= \sigma_y v \end{aligned}$$

Eq. 33 becomes

$$(35) \quad p(R, \sigma_x, \sigma_y) = \frac{1}{2\pi} \iint_{(u^2/a^2 + v^2/b^2) \leq 1} e^{[-(u^2+v^2)/2]} du dv$$

where

$$(36) \quad \begin{aligned} a &= R/\sigma_x \\ b &= R/\sigma_y \end{aligned}$$

Eq. 35 is similar to Eq. 14 and may be evaluated in the same way. Hence

$$(37) \quad p(R, \sigma_x, \sigma_y) = p\left[\frac{R}{2}\left(\frac{1}{\sigma_x} + \frac{1}{\sigma_y}\right), \frac{R}{2}\left(\frac{1}{\sigma_x} - \frac{1}{\sigma_y}\right)\right] - p\left[\frac{R}{2}\left(\frac{1}{\sigma_x} - \frac{1}{\sigma_y}\right), \frac{R}{2}\left(\frac{1}{\sigma_x} + \frac{1}{\sigma_y}\right)\right]$$

INTEGRAL OF THE ELLIPTIC GAUSSIAN DISTRIBUTION OVER A CENTERED ELLIPSE

General forms of this integral are given by

$$(38) \quad \begin{aligned} p(a, b, \rho, \sigma_x, \sigma_y) &= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \iint_{(x^2/a^2 + y^2/b^2) \leq 1} e^{\{-[1/2(1-\rho^2)](x^2/\sigma_x^2 - 2\rho xy/\sigma_x\sigma_y + y^2/\sigma_y^2)\}} dx dy \end{aligned}$$

and

$$(39) \quad p(a, b, \sigma_x, \sigma_y) = \frac{1}{2\pi} \iint e^{-(x^2/2\sigma_x^2 + y^2/2\sigma_y^2)} dx dy$$

where the domain of integration is defined by

$$(40) \quad Ax^2 + Bxy + Cy^2 \leq D; \quad B^2 - 4AC < 0$$

These general forms can always be transformed into Eq. 14 by suitable linear transformations. The transformations are available in standard texts in analytical geometry and are not discussed here.

INTEGRAL OF THE ELLIPTIC GAUSSIAN DISTRIBUTION OVER A CENTERED RANDOMLY ORIENTED ELLIPSE

Returning to Eq. 33, $p(R, \sigma_x, \sigma_y)$ is the probability for given σ_x and σ_y that a missile will hit within a distance, R , of the center of

impact. Hence $p(R, \sigma_x, \sigma_y)$ is the probability distribution in R . If the elliptic distribution has random orientation, all orientations being equally probable, the distribution with respect to R remains the same, but a circular non-Gaussian distribution in hits is obtained. Such a distribution would obtain for a randomly oriented line of delivery with fixed center of impact, or for a rotating target plane where the center of rotation and center of impact are coincident and all orientations of the target plane with respect to any given hit are equally probable.

It is of interest to determine the probability density as a function of the distance, R , from the center of impact. This may be done by expressing $p(R, \sigma_x, \sigma_y)$ in Eq. 37 as a function of R by means of Eq. 5 and 36, differentiating with respect to R and making use of Eq. 29. After some tedious manipulation

$$(41) \quad \frac{dp}{dR} = \frac{1}{\sigma_x \sigma_y} e^{[-(1/4)(1/\sigma_x^2 + 1/\sigma_y^2)R^2]} I_0 \left[\frac{1}{4} \left(\frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2} \right) R^2 \right] R$$

is obtained. An alternative method of deriving Eq. 41 which will be useful in further developments is given in the Appendix.

If, as previously discussed, the elliptic distribution has random orientation, all orientations being equally probable, the probability density with respect to R as given by Eq. 41 remains the same, but becomes uniformly distributed over the area $2\pi R dR$ of arbitrary angular elements of radius R . Then the probability density at the point (R, θ) or (x, y) is given by

$$(42) \quad \frac{1}{2\pi R} \frac{dp}{dR} = \frac{1}{2\pi \sigma_x \sigma_y} e^{[-(1/4)(1/\sigma_x^2 + 1/\sigma_y^2)R^2]} I_0 \left[\frac{1}{4} \left(\frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2} \right) R^2 \right]$$

$$= \frac{1}{2\pi \sigma_x \sigma_y} e^{[-(1/4)(1/\sigma_x^2 + 1/\sigma_y^2)(x^2 + y^2)]} I_0 \left[\frac{1}{4} \left(\frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2} \right) (x^2 + y^2) \right]^*$$

* Steve Gaspar, formerly of this Station, in an internal memorandum dated 30 March 1959 also derived the above probability density function. His derivation differs from those given in this report in that he derives the probability density function directly by considering the probability density at an arbitrary point x, y on the basis of a centered but randomly oriented elliptic Gaussian distribution.

If a centered elliptic target is allowed to have any orientation with respect to an elliptic Gaussian distribution where all orientations of the target are equally probable, the probability of a missile hitting the target under these assumed conditions would be the same as if the orientation of the target were fixed and the orientation of the distribution with respect to the target were random, all orientations being equally probable. The probability density of missile dispersions under this condition is given by Eq. 42. Then the probability of a missile hitting the target is given by

(43)

$p(a, b, \sigma_x, \sigma_y)$

$$= \frac{1}{2\pi\sigma_x\sigma_y} \iint_{(x^2/a^2 + y^2/b^2) \leq 1} e^{[-(1/4)(1/\sigma_x^2 + 1/\sigma_y^2)(x^2 + y^2)]} I_0 \left[\frac{1}{4} \left(\frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2} \right) (x^2 + y^2) \right] dx dy$$

$$= \frac{1}{2\pi\sigma_x\sigma_y} \int_0^{2\pi} \int_0^{t(\theta)} e^{[-(1/4)(1/\sigma_x^2 + 1/\sigma_y^2)t^2]} I_0 \left[\frac{1}{4} \left(\frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2} \right) t^2 \right] t dt d\theta$$

where the second form of Eq. 43 is obtained by means of the transformation of Eq. 15 and $t(\theta)$ is given by Eq. 17. If in Eq. 56 of the Appendix, R is replaced by $t(\theta)$

$$(44) \quad p \left[\frac{t(\theta)}{2} \left(\frac{1}{\sigma_x} + \frac{1}{\sigma_y} \right), \frac{t(\theta)}{2} \left(\frac{1}{\sigma_x} - \frac{1}{\sigma_y} \right) \right] - p \left[\frac{t(\theta)}{2} \left(\frac{1}{\sigma_x} - \frac{1}{\sigma_y} \right), \frac{t(\theta)}{2} \left(\frac{1}{\sigma_x} + \frac{1}{\sigma_y} \right) \right]$$

$$= \frac{1}{\sigma_x\sigma_y} \int_0^{t(\theta)} e^{[-(1/4)(1/\sigma_x^2 + 1/\sigma_y^2)t^2]} I_0 \left[\frac{1}{4} \left(\frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2} \right) t^2 \right] t dt$$

is obtained. Substituting Eq. 44 in Eq. 43,

(45) $p(a, b, \sigma_x, \sigma_y)$

$$= \frac{1}{2\pi} \int_0^{2\pi} p \left[\frac{t(\theta)}{2} \left(\frac{1}{\sigma_x} + \frac{1}{\sigma_y} \right), \frac{t(\theta)}{2} \left(\frac{1}{\sigma_x} - \frac{1}{\sigma_y} \right) \right] - p \left[\frac{t(\theta)}{2} \left(\frac{1}{\sigma_x} - \frac{1}{\sigma_y} \right), \frac{t(\theta)}{2} \left(\frac{1}{\sigma_x} + \frac{1}{\sigma_y} \right) \right] d\theta$$

$$= \frac{2}{\pi} \int_0^{\pi/2} p \left[\frac{t(\theta)}{2} \left(\frac{1}{\sigma_x} + \frac{1}{\sigma_y} \right), \frac{t(\theta)}{2} \left(\frac{1}{\sigma_x} - \frac{1}{\sigma_y} \right) \right] - p \left[\frac{t(\theta)}{2} \left(\frac{1}{\sigma_x} - \frac{1}{\sigma_y} \right), \frac{t(\theta)}{2} \left(\frac{1}{\sigma_x} + \frac{1}{\sigma_y} \right) \right] d\theta$$

due to the periodicity of the integrand.

Equation 45 gives the probability for given σ_x and σ_y of a missile hit on an elliptical target of semi-axes a and b if center of target and center of impact were coincident but the orientation of the target with respect to the distribution were random, all orientations being equally probable. Such a

probability would obtain for a missile hitting a rotating elliptical target where the center of the target, center of rotation, and center of impact were coincident and all orientations of the target with respect to any given hit were equally probable. Equation 45 might find application, for instance, in evaluating the probability of a hit with a depth charge on a submarine (approximated by an ellipse) whose location was known but orientation unknown. Computation of the probability of a hit where for fixed line of delivery the lateral and range deviations are correlated would involve a preliminary rotational transformation of coordinates to present the distribution in terms of uncorrelated variables. Several particular cases of Eq. 45 should be noted. If the target is a circle, then $a = b = R$, $t(\theta) = R$ and Eq. 45 reduces to Eq. 37, the probability that the missile will hit within a distance R of the center of impact as would be expected. If $\sigma_x = \sigma_y = \sigma$, say, Eq. 45 becomes

$$\begin{aligned}
 (46) \quad p(a, b, \sigma) &= \frac{1}{2\pi} \int_0^{2\pi} p\left(\frac{t(\theta)}{\sigma}, 0\right) - p\left(0, \frac{t(\theta)}{\sigma}\right) d\theta \\
 &= \frac{1}{2\pi} \int_0^{2\pi} 1 - e^{-[t^2(\theta)/2\sigma^2]} d\theta \\
 &= \frac{1}{2\pi} \int_0^{2\pi} 1 - e^{-(1/2)a^2b^2/(a^2\sin^2\theta + b^2\cos^2\theta)} d\theta \\
 &= 1 - \frac{1}{2\pi} \int_0^{2\pi} e^{-(1/2)(a/\sigma)^2(b/\sigma)^2/[(a/\sigma)^2\sin^2\theta + (b/\sigma)^2\cos^2\theta]} d\theta
 \end{aligned}$$

where substitutions in the integrand are made from Eq. 6 and 8. The last form of Eq. 46 is similar to that of Eq. 18. Proceeding in the same manner, Eq. 45 reduces to Eq. 32, the probability of the missile hitting a centered elliptical target assuming circular Gaussian distribution. This is consistent, for with this distribution, orientation of the target and/or distribution becomes immaterial. Finally, if $\sigma_x = \sigma_y = \sigma$ and $a = b = R$, Eq. 45 reduces to Eq. 6, the probability of the missile hitting a circular target assuming circular Gaussian distribution.

Equation 45 may be evaluated by numerical integration using tabulated values of the circular coverage function and suitable increments of θ . If Eq. 45 becomes of sufficient importance to merit tabulation, it may be non-dimensionalized as follows. Let

$$(47) \quad k = b/a$$

$$(48) \quad l = \sigma_y/\sigma_x$$

$$(49) \quad m = a/\sigma_y$$

Then Eq. 45 becomes

$$(50) \quad p(a, b, \sigma_x, \sigma_y) = \frac{2}{\pi} \int_0^{\pi/2} p\left[\psi(\theta) \frac{k(l+1)m}{2}, \psi(\theta) \frac{k(l-1)m}{2}\right] \\ - p\left[\psi(\theta) \frac{k(l-1)m}{2}, \psi(\theta) \frac{k(l+1)m}{2}\right] d\theta$$

where

$$(51) \quad \psi(\theta) = \frac{1}{\sqrt{\sin^2 \theta + k^2 \cos^2 \theta}}$$

Equation 51 could be tabulated for a range of values of k , l , and m .

CONCLUSIONS

The integral of the circular Gaussian distribution over a centered ellipse may be directly evaluated from tables of the circular coverage function. The integral of the elliptic Gaussian distribution over a centered circle may be readily evaluated from tables of the circular coverage function. The integral of the elliptic Gaussian distribution over a centered ellipse may be readily evaluated from tables of the circular Gaussian distribution if the axes of the ellipse are parallel to the axes of the elliptic equ-probability contours. If the axes of the ellipse are inclined, evaluation can still be made after a preliminary transformation which may be laborious. Finally the probable value of the integral of the elliptic Gaussian distribution over a centered but randomly oriented ellipse where all orientations are equally probable may be evaluated by numerical integration of an integral involving the circular coverage function.

Appendix

PROBABILITY DENSITY OF A RANDOMLY ORIENTED ELLIPTIC GAUSSIAN DISTRIBUTION

The probability density of Eq. 38 may also be derived from Eq. 37 by the following method. We first transform Eq. 37 by means of Eq. 15. Then

$$(52) \quad p(R, \sigma_x, \sigma_y)$$

$$= \frac{1}{2\pi\sigma_x\sigma_y} \int_0^R e^{-(1/4)(1/\sigma_x^2 + 1/\sigma_y^2)t^2} t \int_{-\pi}^{\pi} e^{-(1/4)(1/\sigma_x^2 - 1/\sigma_y^2)t^2 \cos 2\theta} d\theta dt$$

Substituting

$$(53) \quad \zeta = 2\theta$$

into Eq. 52

$$(54) \quad p(R, \sigma_x, \sigma_y)$$

$$= \frac{1}{4\pi\sigma_x\sigma_y} \int_0^R e^{-(1/4)(1/\sigma_x^2 + 1/\sigma_y^2)t^2} t \int_{-2\pi}^{2\pi} e^{-(1/4)(1/\sigma_x^2 - 1/\sigma_y^2)t^2 \cos \zeta} d\zeta dt$$

$$= \frac{1}{\pi\sigma_x\sigma_y} \int_0^R e^{-(1/4)(1/\sigma_x^2 + 1/\sigma_y^2)t^2} t \int_0^{\pi} e^{-(1/4)(1/\sigma_x^2 - 1/\sigma_y^2)t^2 \cos \zeta} d\zeta dt$$

due to the periodicity of the integrand. From Bessel function theory (Ref. 13)

$$(55) \quad e^{-(1/4)(1/\sigma_x^2 - 1/\sigma_y^2)t^2 \cos \zeta} = I_0 \left[\frac{1}{4} \left(\frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2} \right) t^2 \right]$$

$$+ 2 \sum_{n=1}^{\infty} (-1)^n I_n \left[\frac{1}{4} \left(\frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2} \right) t^2 \right] \cos n\zeta$$

Substituting Eq. 55 in the integrand of Eq. 54 and integrating

$$(56) \quad p(R, \sigma_x, \sigma_y) = P\left[\frac{R}{2}\left(\frac{1}{\sigma_x} + \frac{1}{\sigma_y}\right), \frac{R}{2}\left(\frac{1}{\sigma_x} - \frac{1}{\sigma_y}\right)\right] - P\left[\frac{R}{2}\left(\frac{1}{\sigma_x} - \frac{1}{\sigma_y}\right), \frac{R}{2}\left(\frac{1}{\sigma_x} + \frac{1}{\sigma_y}\right)\right]$$

$$= \frac{1}{\sigma_x \sigma_y} \int_0^R e^{-(1/4)(1/\sigma_x^2 + 1/\sigma_y^2)t^2} I_0\left[\frac{1}{4}\left(\frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2}\right)t^2\right] dt$$

Differentiating Eq. 56 with respect to R, we obtain the probability density as a function of distance from the center of impact. That is,

$$(57) \quad \frac{dp}{dR} = \frac{1}{\sigma_x \sigma_y} e^{-(1/4)(1/\sigma_x^2 + 1/\sigma_y^2)R^2} I_0\left[\frac{1}{4}\left(\frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2}\right)R^2\right] R$$

From the middle member of Eq. 56, it follows that $p(\infty, \sigma_x, \sigma_y) = 1$. This can also be shown directly from the right hand member, making use of the definite integral (Ref. 13)

$$(58) \quad \int_0^\infty e^{(-a\beta)} I_0(\beta) d\beta = \frac{1}{\sqrt{a^2 - 1}}; \quad a > 1$$

NOMENCLATURE

- a, b Axes of elliptic target area or area of integration.
- $I_n(t)$ Modified Bessel function of the first kind of order n .
- p Probability that a missile will hit the target.
- $p(R, r)$ Circular coverage function. The probability that a missile will hit a circle of radius R if it is aimed at a point a distance r from the center of the circle and if it is subject to a circular Gaussian impact-probability distribution of unit standard deviation.
- $\underline{q}(R, r)$ $1 - p(R, r)$. The probability that a missile will not hit a circle of radius R if it is aimed at a point a distance r from the center of the circle and if it is subject to a circular Gaussian impact-probability distribution of unit standard deviation.
- r Distance of center of target from center of impact.
- R Radius of target.
- t Radius vector coordinate.
- x Lateral deviation of missile from center of impact.
- y Range deviation of missile from center of impact.
- z $x + iy$ where $i = \sqrt{-1}$. Complex variable.

- ρ Coefficient of correlation between lateral and range deviations.
- σ Lateral and range standard deviations (used when $\sigma_x = \sigma_y$).
- σ_x Lateral standard deviation.
- σ_y Range standard deviation.
- θ Polar angle coordinate.

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